Connected Domination in Middle Graph

Meenakshi Vaijinathrao[1] **and Faimida Begum**[2]
[1]Government First Grade College, Chitaguppa,
Dist. Bidar, Karnataka(India).
[2]Government First Grade College, Humnadad-585330
Dist. Bidar, Karnataka(India).
Email: [1]amrutha2782008@gmail.com and [2]fbmaths303@gmail.com.

ABSTRACT

The middle graph of a graph G, denoted by $M(G)$is a graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if they are adjacent edges of G or one is a vertex and other is an edge incident with it. A dominating set D of $M(G)$ is called a connected dominating set of $M(G)$ if the induced subgraph $< D >$ is connected. The minimum cardinality of D is called the connected domination number of $M(G)$ and is denoted by $\gamma_c\,[M(G)]$.

In this paper , we establish the upper and lower bounds on $\gamma_c\,[M(G)]$ and compare with other domination parameters of G and elements of G were obtained.

KEYWORDS:_ Middle graph, Domination Number, Connected Domination number.

1.INTRODUCTION

In this paper we follow the notation of [2]. All graphs considered here are simple and finite. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G.

The notation $\alpha_0(G)(\alpha_1(G))$ is the minimum number of vertices(edges) in a vertex(edge) cover of G. The notation $\beta_0(G)(\beta_1(G))$ is the minimum number of vertices(edges) in a maximal independent set of a vertex(edge) of G.

A set D⊆V of G if every vertex not in D is adjacent to a vertex in D. The domination number of G is denoted by $\gamma(G)$ is the minimum cardinality of a dominating set.

A set F of edges in a graph G is called an edge domination set of $G = (V, E)$ if every edge in E−F is adjacent to atleast one edge in F. The minimum cardinality of edges in edge domination set of G is called edge domination number and denoted by $\gamma'(G)$. The edge domination number was studied by S.L.Mitchell and Hedetnime in [5]. Further, if the induced subgraph $< F >$ is connected, then | F | is a connected edge domination number of G, denoted as $\gamma_c'(G)$.

A dominating set D⊆V(G) is a restrained dominating set of G if every vertex not in D is adjacent to a vertex set in G and to a vertex in V-D. The restrained domination number of G is denoted by $\gamma_r(G)$ is the smallest cardinality of a restrained dominating set of G. The concept of restrained domination of graphs introduced by Domke et.al(1999).See[1].

A dominating set D⊆V(G) is a split dominating set if the induced subgraph $< V - D >$ has more than one component. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set. For details see[4].

The independent domination number $i(G)$ of graph G is the minimum cardinality of an independent dominating set.

A dominating set D of a graph G is a global dominating set if D is also a dominating set of $\bar{G}$.The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set of G.

A dominating set D of a graph G is a strong split dominating set if the induced subgraph $< V - D >$is totally disconnected with at least two vertices. The strong split domination number $\gamma_{ss}(G)$ is the minimum cardinality of a strong dominating set of G. See[4].

The concept Roman domination function (RDF) in a graph $G = (V, E)$ is a function $f{:}{-}{>}\,\{0,1,2\}$ satisfying the condition that every vertex u for which f(u)=0 is adjacent to atleast one vertex of v for which f(v)=2 in G. The weight of a Roman dominating function is the value f(v)=$\sum_{u\in v} f(u)$. The minimum weight of a Roman domination function of a graph G is called a Roman domination number and is denoted by $\gamma_R(G)$.

2.RESULTS

The following theorem gives the relationship between $\beta_0(G)$ and $\alpha_1(G)$ with $\gamma_c\,[M(G)]$.

Theorem 1: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] \leq \beta_0(G) + \alpha_1(G)\ .$$

Proof: Suppose $I = \{e_1, e_1, e_2, e_3 \ldots \ldots . e_n\}$ be the set of all end edges in G, where $K \subseteq E(G) - I$ such that $I \cup K$ be the minimal set of edges which cover all the vertices of G. Hence $|\,I \cup K| = \alpha_1(G)$.

Further, $A = \{v_{1,}v_{2,}v_{3,} \ldots\ldots\ldots\ldots v_p\} \subseteq V(G)$ be the maximum set of vertices such that $\deg(v_i,\ v_j) \geq 2$, and $N(v_i) \cap N(v_j) = x$, $\forall v_i, v_j \in A$ so that x $\in V(G) - A$. Clearly |A|=$\beta_0(G)$. Since $V[M(G)] = V(G) \cup E(G)$, let $S = \{u_{1,}u_{2,}u_{3,} \ldots\ldots\ldots\ldots u_i\}$ be the of vertices sub dividing each edge in $M(G)$. We consider a minimal set of vertices $S_1 = \{u_k/1 \leq k \leq i\} \subseteq S$ such that $N[S_1] = V[M(G)]$. Now if $\forall u_k \in S_1$ forms a connected path in a induced sub graph $< S_1 >$ then $|\,S_1|$=$\gamma_c\,[M(G)]$. Otherwise , if there exists more than one component in $< S_1 >$, then consider a subset $S_1 \subset S - S_1$ so that the induced subgraph $< S_2 \cup\ S_1 >$ from a connected sub graph in $M(G)$.Hence $|S_2 \cup S_1| \leq |A| + |I \cup K|$ which gives, $\gamma_c\,[M(G)] \leq \beta_0(G) + \alpha_1(G)$.

The next theorem gives relationship between $\gamma_c\,[M(G)]$ with $\gamma(G)\ and\ \ \alpha_0(G)$.

Theorem 2: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] \geq\ \gamma(G) +\ \alpha_0(G) - 1.$$

Proof: Let $V_1 = \{v_1, v_2, v_3, \ldots \ldots v_n\} \subseteq V(G)$ be the set of all non end vertices in G. Suppose there exists a minimal set $J = \{v_1, v_2, v_3, \ldots\ldots\ \ v_k\} \subseteq V_1$ such that $N[J] = V(G)$. Then J forms a minimal dominating set of G. Now further consider a minimal set of vertices $A \subseteq\ V(G)$, such that every $e_i\ \in E(G)$ is incident to at least one vertex $v_i \in A$ then $|A| =\ \alpha_0(G)$.

Further, since $V[M(G)] = V(G) \cup E(G)$, let $D = \{s_{1,}\ \ s_{2,}s_{3,} \ldots\ldots\ldots\ldots s_q\}$ bethe set of vertices subdividing each edge in $M(G)$. Now, let $S_1 = \{s_{1,}s_{2,}s_{3,} \ldots\ldots\ldots\ldots s_i\}$ ⊆S, $1 \leq i \leq n$. Suppose $N[S_1] = V[M(G)]$. Clearly, S_1 forms the minimal dominating set of $M(G)$.

Suppose the induced subgraph $< S_1 >$ has only one component, then $\{S_1\}$ itself is aconnected dominating set of G. Otherwise, if the induced subgraph $< S_1 >$ has more than one component, then attach minimum number of vertices $\{w_i\} \in V[M(G)] - S_1$, where $\deg(w_i)\ \geq 2$ which are between the vertices of $\{S - S_1\}$ such that $S_2 = S_1 \cup \{w_i\}$ forms exactly one component in the induced subgraph $< S_2 >$. Clearly, S_2 forms a minimal γ_c -set of $M(G)$. Since the set $\{J\} \subset \{S_2\}$ and $\{A\} \subset \{S_2\}$, then $|S_2| \geq |J| + |A| - 1$. Wich gives,

$$\gamma_c\,[M(G)] \geq\ \gamma(G) +\ \alpha_0(G) - 1$$

The following theorem relates $\gamma_c\,[M(G)]$ in terms of $\gamma^{'}(G)$ and $i(G)$.

Theorem 3: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] \geq \gamma^{'}(G) + i(G).$$

Proof: For any connected graph G, there exists $E_1 = \{e_1, e_1, e_2, e_3 \ldots \ldots . e_m\} \subseteq E(G)$ be the set of edges with maximum edge degree and $E_2 = \{e_1, e_1, e_2, e_3 \ldots \ldots . e_n\} \subseteq E(G)$ be the set of edges with minimum edge degree. Suppose $E_1^{'} \subseteq E_1$ and $E_2^{'} \subseteq E_2$, if every edge in $\{E_1^{'}\ \cup E_2^{'}\}$ is adjacent to an edge in$\{V(G) - (E_1^{'}\ \cup E_2^{'})\}$,then $\{E_1^{'}\ \cup E_2^{'}\}$form a$\gamma^{'}(G)$ -set of G. Let $D = \{v_1, v_2, v_3, \ldots \ldots v_m\} \subseteq V(G)$ be the independent set of G, such that $\forall u, v \in V(G), N(u) \cap \{v\} = \varphi$. Suppose N[V]=V(G). Then D is a minimal independent dominating set of G.

Suppose, $= \{u_{1,}\ \ u_{2,}u_{3,} \ldots\ldots\ldots\ldots u_n\} = V[M(G)]$. Further, there exists $C_1 \subseteq C$ such that $u_i \in C_1$ is adjacent to atleast one vertex of $V[M(G)] - C_1$. So that N[C_1]= $V[M(G)]$. Hence C_1 is a minimal dominating set of $M(G)$. Now if $u_i \in C_1$ forms a connected path in the induced subgraph $< C_1 >$ then $|C_1| = \gamma_c\,[M(G)]$. Otherwise, let $C_2 \subseteq\ V[M(G)] - C_1$ and $C_2 \in$ N[C_1].

Consider $C_2^{'} \subseteq C_2$, such that $C_2^{'} \cup C_2$ forms a minimal connected path in $M(G)$,then $\{C_2^{'} \cup C_1\}$ is a minimal connected dominating set in $M(G)$ with $|C_2^{'} \cup C_1| = \gamma_c\,[M(G)]$.

It follows that,

$$|C_2^{'}\ \cup C_1| \geq |E_1^{'}\ \cup E_2^{'}| + |D| \text{ gives}$$

$$\gamma_c\,[M(G)] \geq \gamma'(G) + i(G).$$

A dominating set $D \subseteq V(G)$ is connected dominating set if the induced subgraph $< D >$ has one component. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set of G.

The next theorem gives the relationship between $\gamma_c\,[M(G)]$ and $\gamma_c(G)$.

Theorem 4: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] \geq \gamma_c(G)+1.$$

Proof: Let $S = \{u_{1,}\, u_{2,} u_{3,} \ldots\ldots\ldots\ldots\, u_n\} \subseteq V(G)$ be the minimal set of vertices which cover all the vertices in G. Clearly, S forms a minimal dominating set of G. If the induced subgraph $< S >$ has one component then it self is $\gamma_c(G)$-set.

Further, $S_1 = \{u_{1,}\, u_{2,} u_{3,} \ldots\ldots\ldots\ldots\, u_i\}$ be the vertices subdividing each edge of G in $M(G)$. Since $\deg(u_i) > \deg(u'_i)$, $\forall u'_i \in S$ and $u_i \in S_1$, we consider a minimal set of vertices $S_2 = \{u_i/1 \leq k \leq i\} \subseteq S_1$ such that N$[S_2] = V[M(G)]$. Now if $\forall u_k \in S_2$ forms a connected path in the induced subgraph $< S_2 >$ then $|S_2| = \gamma_c\,[M(G)]$. Otherwise, let $S_3 \subseteq S_1 - S_2$ and $S_3 \in N(S_1)$. consider a set $S'_3 \subseteq S_3$ such that $S_2 \cup S'_3$ forms a minimal connected path in $M(G)$. Thus $\{S_2 \cup S'_3\}$ is the minimal connected dominating set in $M(G)$.

Since the set $\{S\} \subseteq \{S_2 \cup S'_3\}$ then

$|S_2 \cup S'_3| \geq |S| + 1$ which gives,

$$\gamma_c\,[M(G)] \geq \gamma_c(G)+1.$$

Theorem 5: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] + 1 \geq \beta_1(G) + \gamma_s(G).$$

Proof: Suppose $B = \{\, e_1, e_2, e_3 \ldots \ldots . e_n\} \subseteq E(G)$be the maximal set of edges with $N(e_i) \cap N\big(e_j\big) = e$, for every $e_i, e_j \in B, 1 \leq i,j \leq n$ and $e \in E(G) - B$. Clearly, B forms a maximal independent edge set of G. Now consider $F = \{v_1, v_2, v_3, \ldots\ldots\, v_n\}$ be a maximal dominating set of G, if the induced subgraph $< V(G) - F >$ is disconnected, then clearly F forms a split dominating set of G. Since $V[M(G)] = V(G) \cup E(G)$, let $D = \{u_{1,} u_{2,} u_{3,} \ldots\ldots\ldots\ldots\, u_q\}$ be the set of vertices subdividing each edge of G in $M(G)$. Further, $D_1 = \{v_1, v_2, v_3, \ldots\ldots v_i\} \subseteq D, 1 \leq i \leq n$ be the vertices subdividing each edges $e_i \in$ B. $F_1 \subseteq F$ be the set of vertices which are incident with edges $e_1, e_2, e_3 \ldots \ldots . e_j$ such that $N[D_1 \cup F_1] = V[M(G)]$. Clearly, $\{D_1 \cup F_1\}$ is a connected dominating set of $M(G)$. It follows that,

$|D_1 \cup F_1| + 1 \geq |B| + |F|$ and get

$$\gamma_c\,[M(G)] + 1 \geq \beta_1(G) + \gamma_s(G).$$

A dominating set D of a graph G is a cototal dominating set if the induced subgraph $< V - D >$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ is the minimum cardinality of a cototal dominating set of G.

The following theorem relates the cototal domination number of G and between $\gamma_c\,[M(G)]$.

Theorem 6: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] \geq \gamma_{cot}(G) + 1.$$

Proof: Suppose $D = \{v_1, v_2, v_3, \ldots\ldots v_i\} \subseteq$ V(G), such that $N[D] = V(G)$. Then D is a dominating set of G and if the induced subgraph $< V - D >$ has no isolates. Then {D} itself is a $\gamma_{cot}(G)$ −set.

Since $V[M(G)] = V(G) \cup E(G)$, then $J = \{u_{1,} u_{2,} u_{3,} \ldots\ldots\ldots\ldots\, u_i\}$ be the set of vertices subdividing each edge of G in $M(G)$. Now, let $J_1 = \{u_k/1 \leq k \leq i\} \subseteq J$ be the set of vertices subdividing each edge which are incident to each vertex $v_i \in D$ in $M(G)$. Clearly, $N[J_1] = V(G) \subset V[M(G)]$ *and also* $N[J_1] = V(J - J_1)$. Hence $N[J_1] = V(G) \cup V(J - J_1) = V[M(G)]$.Thus the induced subgraph $< J_1 >$ forms a minimal dominating set in $M(G)$. Further, $u_{1,} u_{2,} u_{3,} \ldots\ldots\ldots\ldots\, u_i$ forms a connected path, then $\{J_1\}$ itself forms the minimal connected dominating set in $M(G)$. Otherwise, we consider a set $J_2 \subseteq J - J_1$and $J_2 \in N(J_1)$. Now ,if $J'_2 \subseteq J_2$ such that the induced subgraph $< J'_2 \cup J_1 >$ is a minimal connected subgraph in $M(G)$. Hence, $J'_2 \subseteq J_1$ forms the minimal connected dominating set in $M(G)$. Thus,

$$|J'_2 \cup J_1| \geq |D| + 1$$

$$\gamma_c\,[M(G)] \geq \gamma_{cot}(G) + 1.$$

Theorem 7: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] \geq \gamma_g(G).$$

Proof:Let $D = \{u_{1,}\ u_{2,}u_{3,}............ u_n\} \subseteq V(G)$ and $D \subseteq V(\overline{G})$. If $N[D] = V(G)$and $N[D] = V(\overline{G})$ .Then D is a global dominating set of G. Further,$V[M(G)] = V(G) \cup E(G)$. Let $S = \{u_{1,}u_{2,}u_{3,}............ u_i\}$ be the set of vertices subdividing each edge of G in $M(G)$. $S_1 = \{u_k/1 \leq k \leq i\} \subseteq S$ be the set of vertices subdividing each edge e_k which are incident the vertices of vertex set of S_1 in $M(G)$. Clearly, $N[S_1] = V(G)$ and also in M(G), $N[S_1] = V(S - S_1) = V[M(G)]$. Thus$\{S_1\}$ forms a minimal dominating set in M(G). Further, if $u_{1,}\ u_{2,}u_{3,}............ u_i$ forms a connected path in the induced subgraph $< S_1 >$ in M(G). Then S_1 is a γ_c $-$set of M(G). Otherwise, consider a set $S_2 \subseteq S - S_1$ and $\forall v_j \in S_2 \in N(S_1)$. Consider a subset $S_2' \subseteq S_2$.So that $\{S_2' \cup S_1\}$ form a single component in the induced subgraph $< S_2' \cup S_1 >$. Then $\{S_2' \cup S_1\}$ forms a minimal dominating set of $M(G)$. Thus, $|S_2' \cup S_1| \geq |D|$, which gives.

$$\gamma_c\,[M(G)] \geq \gamma_g(G).$$

A dominating set of D is a weak dominating set of G if for every vertex $u \in V(G) - D$ there is vertex $v \in D$ with $\deg(v) \leq \deg(u)$ and u is adjacent to v see[3].

Theorem 8: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] \leq diam(G) + \gamma_w(G) \text{ and } G \neq W_n; n \geq 6.$$

Proof: Suppose $G = W_n$ with $n \geq 6$. Then $\gamma_c\,[M(G)] \nleq diam(G) + \gamma_w(G)$. Suppose $F = \{u_{1,}\ u_{2,}u_{3,}............ u_k\} \subseteq V(G)$ be the set of vertices $u \in V(G) - F$ is adjacent with atleast one vertex $u \in F$ and $\deg(v) \leq \deg(u)$.F is a dominating set of G. Further, $S = \{e_1, e_2, e_3 \ldots \ldots . e_j\}$ be the minimal set of edges which constitute the longest path between any two distinct vertices $u, v \in V(G)$ such that dist(u,v)=diam(G). Clearly, $|S| = \text{diam}(G)$. Since $V[M(G)] = V(G) \cup E(G)$, let $S \subseteq V(G)$. Now, let $D = \{s_{1,}\ s_{2,}s_{3,}............ s_i\}$ be the set of vertices sub dividing each edge in $M(G)$. Each $D_1 = \{s_q/1 \leq q \leq i\}$ be the set of vertices subdividing each edge $e_1, e_2, e_3 \ldots \ldots . e_q$ in $M(G)$ and $D_2 = \{s_q'/1 \leq q \leq i\}$ be the set of vertices subdividing the edges $E(G) - S$ in $M(G)$.Clearly,$N(D_1) = F \cup V(D_2)$. Further, consider a minimal set $D_2' \subseteq D_2$such that $N[D_2' \cup D_1] = V[M(G)]$. Then $D_3 = D_2' \cup D_1$ forms the minimal dominating set in $M(G)$. If the induced subgraph $< D_3 >$is minimal connected subgraph of $M(G)$,then $\{D_3\}$ itself is the minimal connected dominating set $M(G)$. Otherwise, let $D_3' \subset V[M(G)] - D_3$ and $D_3' \in N(D_3)$. Now if $D_3'' \subset D_3'$ such that $\{D_3'' \cup D_3\}$ is the minimal dominating set in $M(G)$. If the induced subgraph $< D_3'' \cup D_3 >$ is the minimal connected subgraph of $M(G)$. Since $S \subset M(G)$,

we have, $|D_3'' \cup D_3| \leq |S| + |F|$, so that

$$\gamma_c\,[M(G)] \leq diam(G) + \gamma_w(G).$$

A dominating set D⊆V is a restrained dominating set of G if every vertex not in D is adjacent to a vertex in D and to a vertex in V – D. The restrained domination number of G is denoted by $\gamma_r(G)$ is the smallest cardinality of a restrained dominating set of graphs introduced by Domke et.al(1999) see[1].

Theorem 9: For any connected (p,q) graph G,

$$\gamma_c\,[M(G)] \geq \gamma_r(G)+1.$$

Proof: Consider a set $F = \{v_1, v_2, v_3, \ldots \ldots v_p\} \subseteq V(G)$ be the set of end vertices in G and $B' = V(G) - B$. Then there exists a vertex set $H \subseteq B'$ such that $\forall v_i \in \{V(G) - \{H \cup B\}\}$ is adjacent to at least one vertex of $\{H \cup B\}$ and in $V(G) - \{H \cup B\}$ is a γ_r $-$set fo G. Since $V[M(G)] = V(G) \cup E(G)$,then $I = \{u_{1,}u_{2,}u_{3,}............ u_i\}$ be the set of vertices subdividing each edge of G in $M(G)$. Now ,let $I_1 = \{u_k/1 \leq k \leq i\} \subseteq I$ be the set of vertices subdividing each edge which are incident to each vertex $v_i \in \{H \cup B\}$ in $M(G)$. Clearly,$N[I_1] = V(I - I_1)$.Hence $N[I_1] = V(G) \cup V(I - I_1)$= $V[M(G)]$. Thus the induced subgraph $< I_1 >$ forms a minimal dominating set in $M(G)$. Further, $u_{1,}u_{2,}u_{3,}............ u_i$ form a connected path, then $\{I_1\}$ itself forms the minimal connected dominating set in $M(G)$. Otherwise, we consider a set $\{I_2\} \subseteq I - I_1$ and $\forall v_i \in I_2$ are neighbours of I_1. Now, if $I_2' \subseteq I_2$ such that the induced subgraph $< I_2' \cup I_1 >$ is minimal connected subgraph in $M(G)$. Hence $\{I_2' \cup I_1\}$ forms the minimal connected dominating set in $M(G)$.

Thus, $|I_2' \cup I_1| \geq |H \cup B|$+1
$\gamma_c\,[M(G)] \geq \gamma_r(G)$+1.

Theorem 10: For any connected (p,q) graph G,
$\gamma_c\,[M(G)] \geq \Delta(G) + \gamma(G) - 1$.

Proof: For any connected graph G with non end vertices, there exist atleast one vertex $v \in V(G)$ such that, $\deg(v) = \Delta(G)$. Let $C = \{v_1, v_2, v_3, \ldots\ldots\, v_n\} = V(G)$. Suppose there exists $C_1 \subseteq C$ such that $\forall v_i \in C_1$ is adjacent to atleast one vertex of $V(G) - C_1$, so that $N[C_1] = V(G)$. Then C_1 is a minimal dominating set of G. Let $S = \{u_{1,}u_{2,}u_{3,}$............ $u_q\}$ be the set of vertices subdividing each edge of G in $M(G)$. $S_1 = \{u_q/1 \leq q \leq i\}$ bethe set of vertices subdividing the edges which are incident with the maximum degree vertex v in G. Also, $S_2 = \{u_q/1 \leq q \leq j\}$ be the set of vertices subdividing each edge e_i of G, which are incident with vertices $v_i \in C_1$. Now if $S_1' \subseteq S_1$and $S_2' \subseteq S_2$ such that the induced subgraph $< S_1' \cup S_2' >$ is the minimal connected subgraph of $M(G)$, then $\{S_1' \cup S_2'\}$ itself is the minimal connected dominating set in $M(G)$.

This follows that,
$\{S_1' \cup S_2'\} \geq \Delta(G) + |C_1| - 1$ and gives
$\gamma_c\,[M(G)] \geq \Delta(G) + \gamma(G) - 1$.

In the following theorem we establish relation between Roman domination number $\gamma_R(G)$ and edge connected domination number $\gamma_c'(G)$ with connected domination in middle graph.

Theorem 11: For any connected (p,q) graph G,
$\gamma_c\,[M(G)] \leq \gamma_R(G) + \gamma_c'(G)$ and $G \neq K_{1,n}; n \geq 4$.

Proof: If $G = K_{1,n}$ with $n \geq 4$ then $\gamma_c\,[M(G)] \nleq \gamma_R(G) + \gamma_c'(G)$. Suppose A= $\{ e_1, e_2, e_3 \ldots \ldots e_i\}$ be the minimal edge dominating set of G and if induced subgraph does not contain more than one component. Then A itself is a connected edge dominating set of G. Otherwise, if the induced subgraph $< A >$ has more than one component, then attach the minimum number of edges $\{e_k\} \in E(G) - A$ which are in every path of $E(G) - A$ such that $A_1 = A \cup \{e_k\}$ forms exactly one component. Clearly, A_1forms a γ_c'- set of G. Suppose the function $f: V(G) \rightarrow \{0,1,2\}$and partition the vertex set $V(G)$ into (V_0, V_1, V_2) induced by f with $|V_i| = n_i$ for i=0,1,2. Suppose the set V_2 dominated the V_0. Then $B = V_1 \cup V_2$ forms a minimal Roman dominating set of G.

Further, since $V[M(G)] = V(G) \cup E(G)$, let $D = \{u_{1,}u_{2,}u_{3,}$............ $u_i\}$ be the set of vertices subdividing each edge of G in $M(G)$. Now, consider the set $D_1 = \{u_k/1 \leq k \leq i\} \subseteq D$ be the set of vertices subdividing each edge $e_k \in A$, $1 \leq k \leq i$ in $M(G)$. Clearly,$N[D_1] = V(G)$ and also in $M(G)$. Since, $N[D_1] = V(D - D_1)$. Then $N[D_1] = V(G) \cup V(D - D_1) = V[M(G)]$. Thus the induced subgraph $< D_1 >$ forms a minimal dominating set of $M(G)$. Further, if $u_{1,}u_{2,}u_{3,}$............ u_i forms a connected path, then the induced subgraph $< D_1 >$ itself is a minimal connected dominating set of $M(G)$. Otherwise, we consider a set $D_2 \subseteq D - D_1$ and for every element of D_2, so that $D_2 \in N[D_1]$. Now if $D_2' \subseteq D_2$ such that the induce subgraph $< D_2' \cup D_1 >$ is the minimal connected subgraph in $M(G)$, then $\{D_2' \cup D_1\}$ forms the minimal connected dominating set in $M(G)$. Thus,
$|D_2' \cup D_1| \leq |A| + |B|$ which gives
$\gamma_c\,[M(G)] \leq \gamma_R(G) + \gamma_c'(G)$.

Theorem 12: For any connected (p,q) graph G,
$\gamma_c\,[M(G)] \geq \gamma_{ss}(G) + 1$.

Proof: For the strong split dominating set $\gamma_{ss}(G)$. We consider a set $S = \{v_1, v_2, v_3, \ldots\ldots v_n\} \subseteq V(G)$, such that for every vertex of $V(G) - S$ is adjacent to atleast one vertex of S and $N[S] = V(G)$. If the induced subgraph $< S >$ is totally disconnected. Then S is a γ_{ss} −sset of G. Since $V[M(G)] = V(G) \cup E(G)$, $S \subseteq V[M(G)]$. Now, let $K = \{u_{1,}u_{2,}u_{3,}$............ $u_i\}$ be the set of vertices subdividing each edge of G in $M(G)$. Let $K_1 = \{u_q/1 \leq q \leq i\}$ be the set of vertices subdividing the edges $e_1, e_2, e_3 \ldots\ldots e_q$ of G in $M(G)$ and $K_2 = \{u_q/1 \leq q \leq i\}$ be the set of vertices subdividing the edges which are incident with the vertices $v_i \in K$ in $M(G)$. Further consider a minimal set $K_2' \subseteq K_2$ such that $N[K_2'] = V[M(G)]$. Then K_2' forms the minimal dominating set in $M(G)$. If the induced subgraph $< K_2' >$ is the minimal connected subgraph of $M(G)$, then $\{K_2'\}$ itself is the minimal dominating set in $M(G)$. Otherwise, let $K_3 \subseteq V[M(G)] - K_2'$ and for every elements of K_3,such that $K_3 \in N[K_2']$. Now if $K_3' \subseteq K_3$and the induced subgraph $< K_3' \cup K_3 >$ is the minimal connected subgraph of $M(G)$, then $\{K_3' \cup K_3\}$ is the minimal dominating set in $M(G)$.

Thus we have, $|K_3' \cup K_3| \geq |S| + 1$

$$\gamma_c\,[M(G)] \geq \gamma_{ss}(G) + 1.$$

Theorem 13: For any connected (p,q) graph G,

$\gamma_c\,[M(G)] \leq p + m - 1$, where 'm' is the number of end vertices in G.

Proof: Consider $F = \{v_1, v_2, v_3, \ldots\ldots v_n\} \subset V(G)$ be the set of all end vertices in G. Since $V[M(G)] = V(G) \cup E(G)$. Now ,let Let $S = \{u_{1,} u_{2,} u_{3,} \ldots\ldots\ldots\ldots u_q\}$ be the set of vertices subdividing each edge of G in $M(G)$.Consider $S_1 = \{u_{1,}\ u_{2,} u_{3,} \ldots\ldots\ldots\ldots u_i\} \subseteq S$ subdividing end edges which are also incident with end vertices of G and also in $M(G)$. Suppose $S_2 = S - S_1$ subdividing edges other than the end edges in G, also in $M(G)$. Suppose $S_3 \subseteq S_2$ and $\{S_1 \cup S_3\}$ gives a connected subgraph in $M(G)$. Then $\{S_1 \cup S_3\}$ forms a minimal connected dominating set in $M(G)$.

It follows that, $\{S_1 \cup S_3\} \leq p + |F| - 1$ so that

$$\gamma_c\,[M(G)] \leq p + m - 1.$$

The following theorem gives the exact value of $\gamma_c\,[M(T)]$ for any tree i terms of the edges of T.

Theorem 14: For any connected (p,q) tree T, $\gamma_c\,[M(T)] = q$.

Proof: Let $A = \{v_1, v_2, v_3, \ldots\ldots v_n\} = V(T)$ and $B = \{e_1, e_2, e_3 \ldots\ldots . e_n\}$ be the set of edges of a tree T. Since $V[M(T)] = A \cup B$.

Now, we consider the following cases.

Case 1:Suppose a tree T is a star. Then the set $\{B\} \cup v$ forms a complete subgraph, whose $deg v = \Delta(T)$ and $\{B\} \subset V[M(T)]$.Since $|B| = N[V(T)]$ and $< B >$ is connected, then B is a connected dominating set of $M(T)$. Hence $|B| = E(T) = q$.

Case 2:Suppose a tree T is a path. Let $P_n = \{v_1, v_2, v_3, \ldots\ldots v_n\} = V(T)$ and $\{e_1, e_2, e_3 \ldots\ldots . e_m\} = E(T)$.

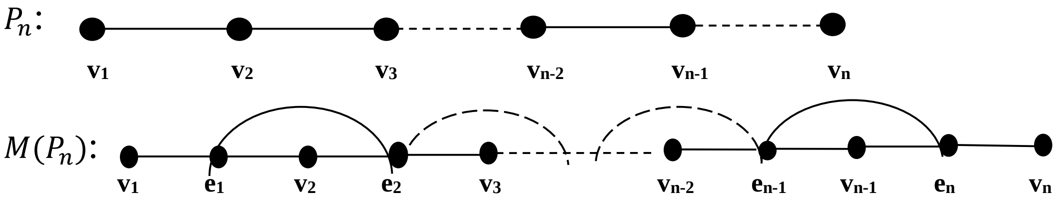

Fig.1.

In middle graph of a path P_n. Let $B = \{e_1, e_2, e_3 \ldots\ldots . e_n\}$ be the set of vertices in $M(P_n)$, so that every vertex of $M(P_n) - B$ is adjacent to at least one vertex of B. Thus B is a dominating set of $M(P_n)$. From the above figure 1, the induced subgraph $< B >$ is connected, then $\{B\}$is a minimal dominating set of $M(P_n)$.

Case3:Suppose neither a tree T is a star nor a path. Let H⊂A be the set of all end vertices in a tree T. Since B⊂ $V[M(T)]$ and $e_i \in B$ is adjacent to atleast one vertex of $V[M(T)] - H$, then $B$is a dominating set of $M(T)$. If the induced subgraph $< B >$ is connected, then $\{B\}$ is a minimal connected dominating set of $M(T)$. Hence $|B| = q$. From the above all the three cases. We come to know that $|B| = q$ and $\{B\}$ is a connected dominating set $M(T)$. Clearly, $\gamma_c\,[M(T)] = q$.

REFERENCES

1]G.S. Domke J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar and L.R.Markus. Restrained domination In graphs, Discrete Mathematics Vol-203,1999,61-69.

2]F. Harary,(1969),Graph theory, Adision Wesley, Reading Mass(61-62) .

3]T.W.Haynes, S.T. Hedetniemi and P.J.Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc: New York(1998).

4]V.R.Kulli, Theory of domination in Graphs, Vishwa International Publications,2010.

5]S.L.Mitchell and S.T. Hedetniemi(1977), Edge domination in trees. In:proc. 8th S.E.Conf. on Combinatorics, Graph theory,utilitas Mathematica, Winnipeg, 19(489-509).